Introduction to Global Navigation Satellite System (GNSS) Position Computation

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Pseudorange equation

Perfect World: $r = c\left(t_R - t^S\right)$

Real World:

Receiver clock error; Tropospheric delay; Multipath

$$\rho = r + c\left(\delta t_R - \delta t^S\right) + I + T + M + \xi$$

Satellite clock error; Ionospheric delay; Thermal noise

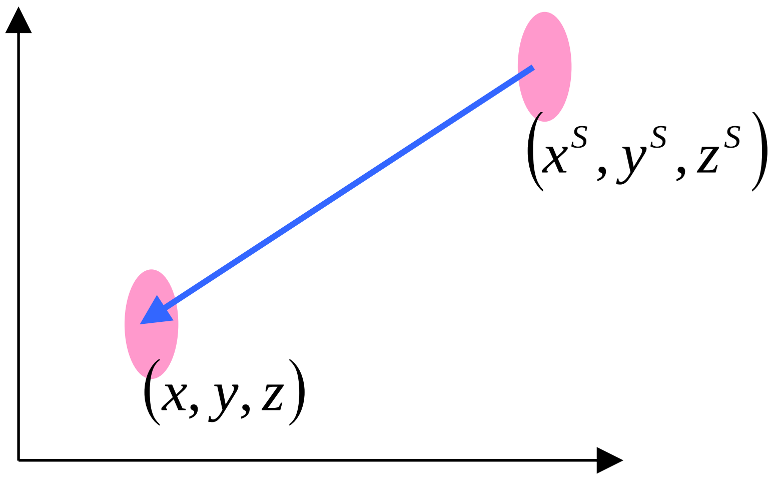
Simplify… $\rho = r + c\left(\delta t_R - \delta t^S\right) + \varepsilon$

Range Equation

Satellite position at signal transmission time: $\left(x^S, y^S, z^S\right)$

Receiver position at signal reception time: $\left(x, y, z\right)$

$$r = \sqrt{\left(x - x^S\right)^2 + \left(y - y^S\right)^2 + \left(z - z^S\right)^2}$$



Pseudorange model

$$\rho = \underbrace{\sqrt{(x - x^S)^2 + (y - y^S)^2 + (z - z^S)^2}}_{r} + c(\delta t_R - \delta t^S) + \varepsilon$$

- (blue circle) Given satellite position & clock in navigation message
- (magenta circle) Unknown receiver position & clock
- (yellow circle) Estimate optimal solution to minimize the error

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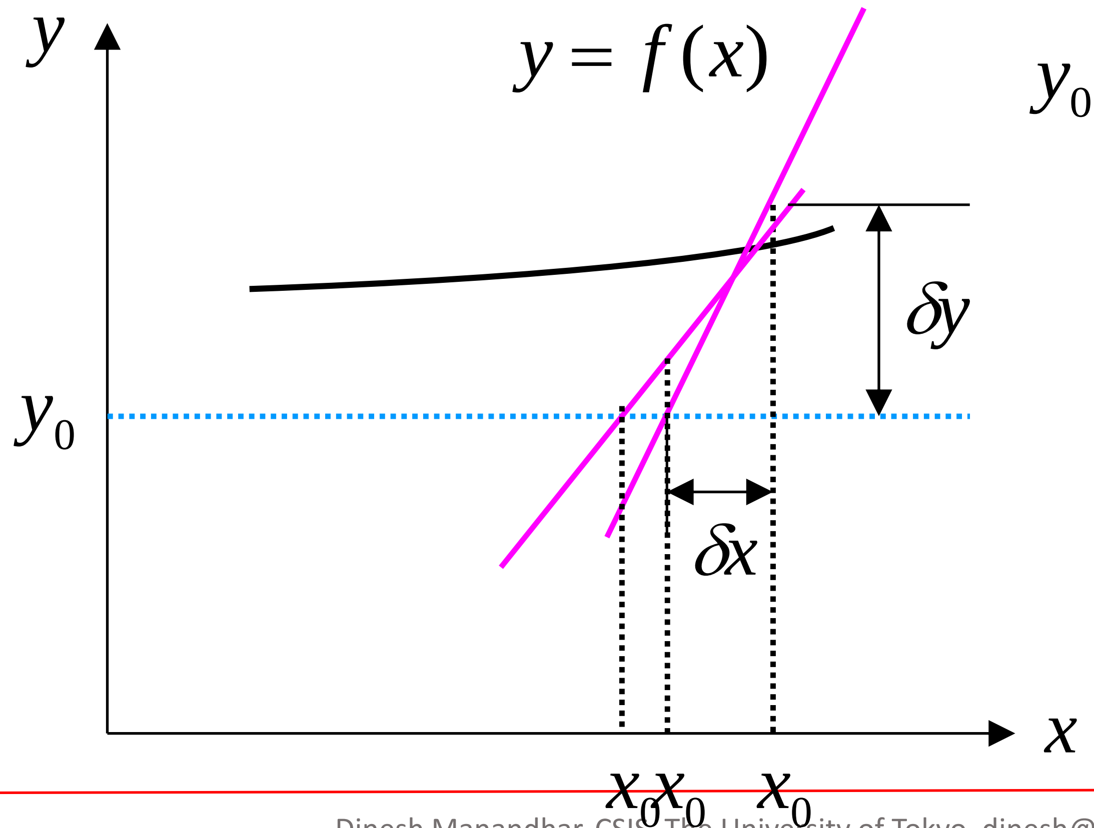
Nonlinear Optimization Problem

- We have *n* simultaneous nonlinear equations from *n* pseudorange observations.
- We need at least 4 independent observations in order to determine 4 unknown parameters.
- In general, even a single nonlinear equation cannot be solved without some iterative method by generating a sequence of approximate solutions.

Newton-Raphson Method

Find successively better approximation $x = x_0$ satisfying $y = y_0$ of a nonlinear equation $y = f(x)$



$$y_0 - f(x_0) = f'(x_0) \cdot \delta x$$

(the term $y_0 - f(x_0)$ is marked δy)

Newton-Raphson Algorithm

Initial Guess $x = x_0$

↓

Solve the equation for δx

$$\underbrace{y - f(x_0)}_{\delta y} = f'(x_0) \cdot \delta x$$

δy: Residual

↓

Update x_0

$$x_0 \leftarrow x_0 + \delta x$$

Repeat until a sufficiently small value δx is reached.

Pseudorange Equation

$$\rho = \underbrace{\sqrt{\left(x - x^S\right)^2 + \left(y - y^S\right)^2 + \left(z - z^S\right)^2}}_{r} + \underbrace{c\delta t}_{b} - c\delta t^S$$

$$= f\left(x, y, z, b\right)$$

For given observation $\rho = \rho_0$
Linearize around the initial solution $\left(x_0, y_0, z_0, b_0\right)$
Obtain the update $\left(\delta x, \delta y, \delta z, \delta b\right)$

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Linearization

Partial derivatives with respect to each unknown parameter:

$$\frac{\partial f}{\partial x} = \frac{x - x^S}{r}, \quad \frac{\partial f}{\partial y} = \frac{y - y^S}{r}, \quad \frac{\partial f}{\partial z} = \frac{z - z^S}{r}, \quad \frac{\partial f}{\partial b} = 1$$

Linearized pseudorange residual equation:

$$\underbrace{\rho_0 - f(x_0, y_0, z_0, b_0)}_{\delta\rho} = \frac{x_0 - x^S}{r_0}\delta x + \frac{y_0 - y^S}{r_0}\delta y + \frac{z_0 - z^S}{r_0}\delta z + \delta b$$

Vector Description

$$\delta\rho = \underbrace{\begin{bmatrix} \dfrac{x_0 - x^S}{r_0} & \dfrac{y_0 - y^S}{r_0} & \dfrac{z_0 - z^S}{r_0} & 1 \end{bmatrix}}_{\mathbf{h}} \underbrace{\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta b \end{bmatrix}}_{\mathbf{x}} + \varepsilon$$

We need at least 4 linearly independent equations in order to determine 4 unknown parameters.

Simultaneous equations

$$\begin{bmatrix} \delta\rho^1 \\ \delta\rho^2 \\ M \\ \delta\rho^N \end{bmatrix} = \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ M \\ \mathbf{h}^N \end{bmatrix} \mathbf{x} + \begin{bmatrix} \varepsilon^1 \\ \varepsilon^2 \\ M \\ \varepsilon^N \end{bmatrix} \qquad N \geq 4$$

$$\underbrace{}_{\mathbf{y}} \quad \underbrace{}_{\mathbf{H}} \quad \underbrace{}_{\mathbf{e}}$$

$$(N \times 1) \quad (N \times 4)\ (4 \times 1) \quad (N \times 1)$$

Residual vector → $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$

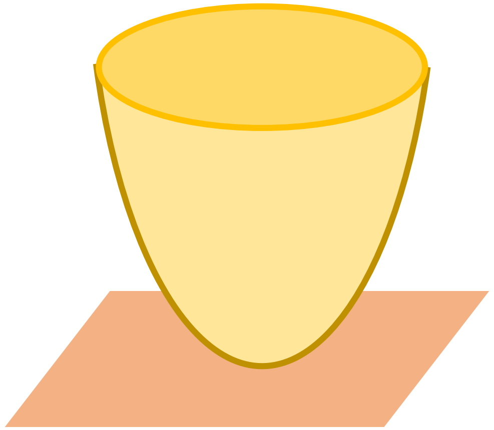
Observation matrix (→ $\mathbf{H}$)   State vector (→ $\mathbf{x}$)

Least squares problem

For given $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$, find $\mathbf{x} = \hat{\mathbf{x}}$ to minimize $\mathbf{e}^T\mathbf{e}$.

Performance Index:

$$
\begin{aligned}
J &= \mathbf{e}^T\mathbf{e} \\
&= (\mathbf{y} - \mathbf{H}\mathbf{x})^T(\mathbf{y} - \mathbf{H}\mathbf{x}) \\
&= \mathbf{y}^T\mathbf{y} - \mathbf{y}^T\mathbf{H}\mathbf{x} - \mathbf{x}^T\mathbf{H}^T\mathbf{y} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x} \\
&= \mathbf{y}^T\mathbf{y} - 2\mathbf{x}^T\mathbf{H}^T\mathbf{y} + \mathbf{x}^T\mathbf{H}^T\mathbf{H}\mathbf{x}
\end{aligned}
$$



Find $\mathbf{x} = \hat{\mathbf{x}}$ to minimize $J \Leftrightarrow \left.\frac{\partial J}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}} = 0$

Least squares solution

Partial derivatives of a scalar function w.r.t the state vector:

(1) $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} = \mathbf{x}^T\mathbf{a}$ then $\dfrac{\partial f}{\partial \mathbf{x}} = \mathbf{a}$

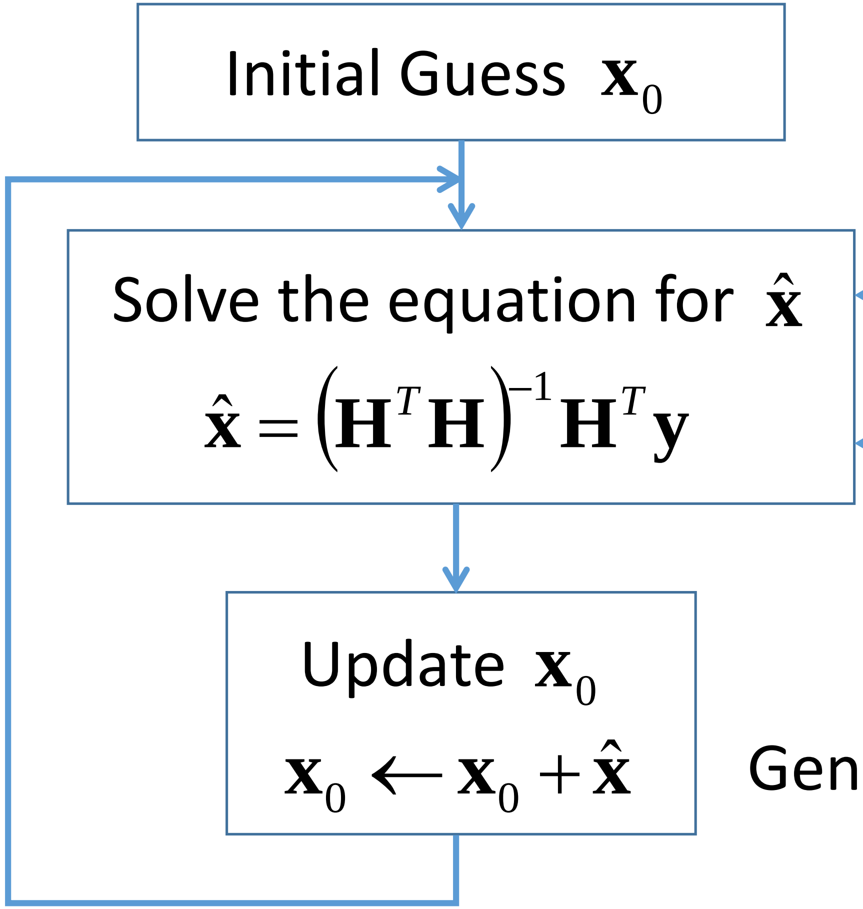
(2) $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ then $\dfrac{\partial f}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x}$

for all symmetric matrix $\mathbf{A}$

Find $\mathbf{x} = \hat{\mathbf{x}}$ to satisfy $\dfrac{\partial J}{\partial \mathbf{x}} = -2\mathbf{H}^T\mathbf{y} + 2\mathbf{H}^T\mathbf{H}\mathbf{x} = \mathbf{0}$

$$\Rightarrow \quad \hat{\mathbf{x}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}$$

GNSS Positioning Calculation



Even the north pole is a good initial guess!

Pseudorange ρ

Satellite position $\left(x^S, y^S, z^S\right)$

Satellite clock δt^S

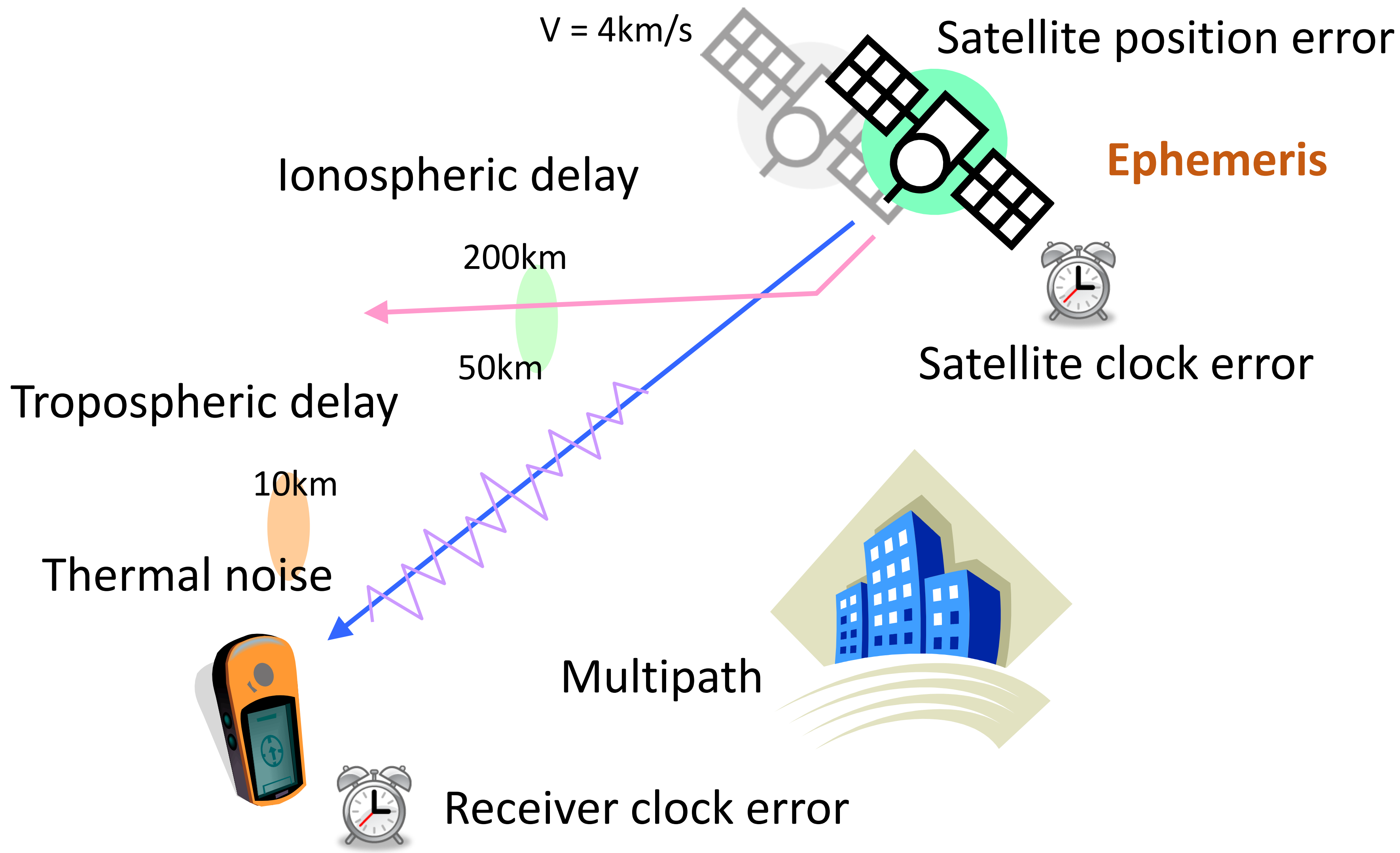
Generally takes a few iterations.

Error Budget

- The positioning accuracy depends on the magnitude of error in the individual pseudorange measurement.

Source	Error	DGPS
Satellite orbit error	1 ~2m	0
Satellite clock error	1 m	0
Ionospheric delay	4~10 m	Can be minimized to <1m
Tropospheric delay	1~2 m	
Thermal noise	1 m	Can't be removed
Multipath	1m or more	

Error sources



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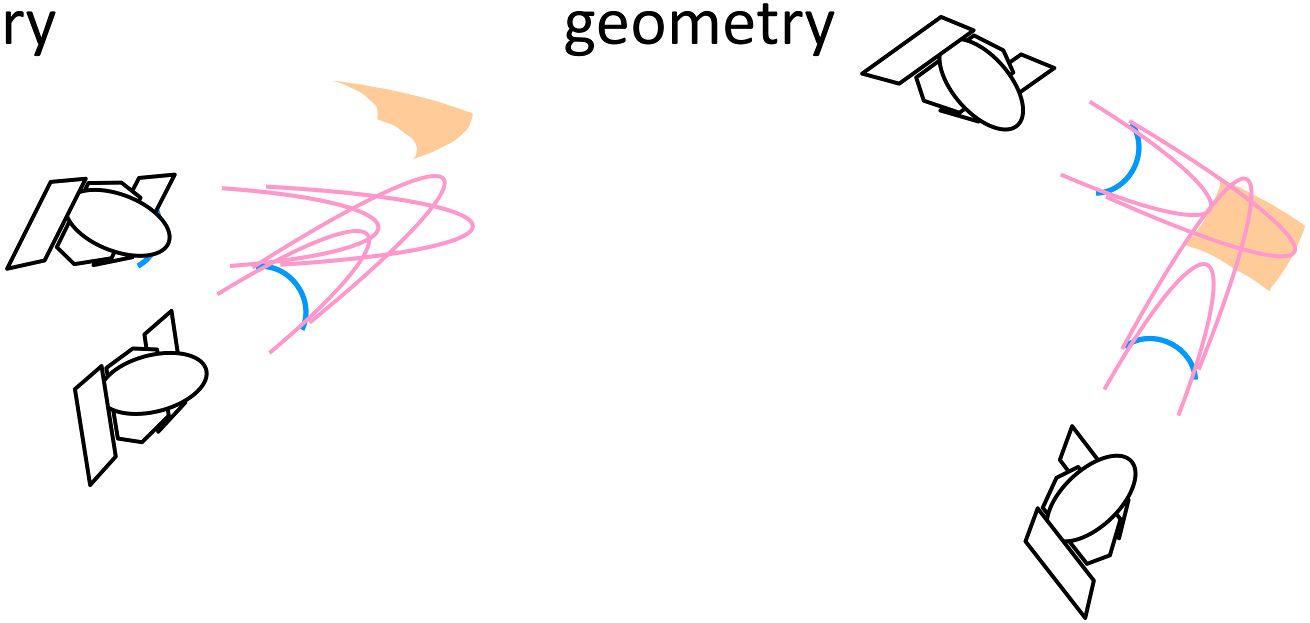
Satellite geometry and positioning error

- The positioning accuracy also depends on the geometric configuration of the satellites.

Bad geometry

Good geometry



Dilution of precision (DOP)

$$\mathbf{G} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1} = \begin{bmatrix} g_x & \cdot & \cdot & \cdot \\ \cdot & g_y & \cdot & \cdot \\ \cdot & \cdot & g_z & \cdot \\ \cdot & \cdot & \cdot & g_b \end{bmatrix}$$

Position DOP: $\mathrm{PDOP} = \sqrt{g_x + g_y + g_z}$

Time DOP: $\mathrm{TDOP} = \sqrt{g_b}$

Geometric DOP: $\mathrm{GDOP} = \sqrt{g_x + g_y + g_z + g_b}$

DOP and positioning accuracy

Accuracy of any measurement is proportionately dependent on the DOP value. This means that if DOP value doubles, the resulting position error increases by a factor of two.